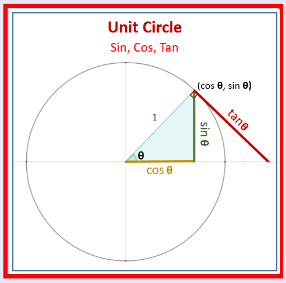
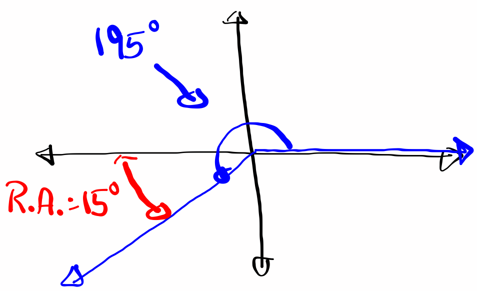
Math 241

Winter 2023

Lecture 12



Find exact value of Cos 195°.



$\cos 195° = -\cos 15°$

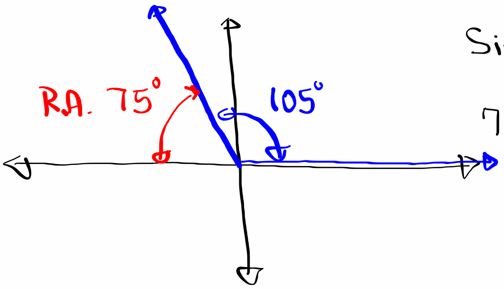
$15° = 45° - 30°$

$\cos 15° = \cos(45° - 30°) = \cos 45° \cos 30° + \sin 45° \sin 30°$

$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \boxed{\frac{\sqrt{6} + \sqrt{2}}{4}}$

$\cos 195° = -\cos 15° = \boxed{-\frac{\sqrt{6} + \sqrt{2}}{4}}$

Find exact Value of Sin 105°



$\sin 105^\circ = \sin 75^\circ$

$75^\circ = 45^\circ + 30^\circ$

$$\sin 75^\circ = \sin(45^\circ + 30^\circ) = \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ$$

$$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \boxed{\frac{\sqrt{6}+\sqrt{2}}{4}}$$

$$\sin 105^\circ = \sin 75^\circ = \boxed{\frac{\sqrt{6}+\sqrt{2}}{4}}$$

Notice

$\sin 75^\circ = \cos 15^\circ$

we have cofunctions with complementary angles.

Find the exact Value of

$$\frac{\tan\frac{\pi}{18} + \tan\frac{\pi}{9}}{1 - \tan\frac{\pi}{18} \cdot \tan\frac{\pi}{9}}$$

Do You recognize that we have

$$\frac{\tan A + \tan B}{1 - \tan A \cdot \tan B}$$

$$= \tan(A + B)$$

$$= \tan\left(\frac{\pi}{18} + \frac{2\cdot\pi}{2\cdot 9}\right)$$

$$= \tan\left(\frac{\pi}{18} + \frac{2\pi}{18}\right)$$

$$= \tan\left(\frac{3\pi}{18}\right) = \tan\frac{\pi}{6} = \tan 30^\circ = \boxed{\frac{\sqrt{3}}{3}}$$

Find x:

$$\sin\left(\frac{\pi}{2} - x\right) = \cos\left(\frac{\pi}{2} + 2x\right)$$

Cofunctions

$$\cancel{\frac{\pi}{2}} - x + \frac{\pi}{2} + 2x = \cancel{\frac{\pi}{2}}$$

$$x + \frac{\pi}{2} = 0 \to \boxed{x = -\frac{\pi}{2}}$$

Simplify

$$\cos\left(x + \frac{\pi}{6}\right) + \sin\left(x - \frac{\pi}{3}\right) =$$

$$\cancel{\cos x \cos\frac{\pi}{6}} - \cancel{\sin x \sin\frac{\pi}{6}} + \cancel{\sin x \cos\frac{\pi}{3}} - \cancel{\cos x \sin\frac{\pi}{3}} =$$

$$\cos\frac{\pi}{6} = \sin\frac{\pi}{3}, \quad \sin\frac{\pi}{6} = \cos\frac{\pi}{3}$$

cofunctions with compl. angles.

$$= \boxed{0}$$

Verify $\dfrac{\sin 3x}{\sin x \cos x} = 4\cos x - \sec x$

Hint: $3x = 2x + x$

$$\frac{\sin 3x}{\sin x \cos x} = \frac{\sin(2x + x)}{\sin x \cos x} = \frac{\sin 2x \cos x + \cos 2x \sin x}{\sin x \cos x}$$

$$= \frac{2\sin x \cos x \cdot \cos x + (\cos^2 x - \sin^2 x)\cdot \sin x}{\sin x \cos x}$$

$$= \frac{2\sin x \cos^2 x + \sin x \cos^2 x - \sin^3 x}{\sin x \cos x}$$

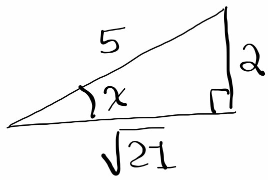
$$= \frac{3\sin x \cos^2 x - \sin^3 x}{\sin x \cos x} = \frac{3\cancel{\sin x}\cos^{\cancel{2}} x}{\cancel{\sin x}\cancel{\cos x}} - \frac{\sin^{\cancel{3}} x}{\cancel{\sin x}\cos x}$$

$$= 3\cos x - \frac{\sin^2 x}{\cos x} = 3\cos x - \frac{1 - \cos^2 x}{\cos x}$$

$$= 3\cos x - \frac{1}{\cos x} + \frac{\cos^{\cancel{2}} x}{\cancel{\cos x}} = 3\cos x - \sec x + \cos x$$

$$= \boxed{4\cos x - \sec x}$$

$\sin x = \frac{2}{5}$, $\underbrace{90^\circ < x < 180^\circ}_{\text{QII}}$, Find $\tan \frac{x}{2}$



Recall

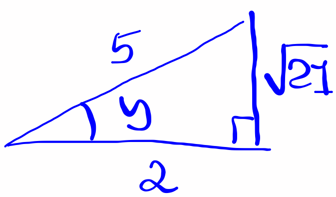
$$\tan \frac{x}{2} = \frac{1-\cos x}{\sin x} = \frac{\sin x}{1+\cos x}$$

$$\tan \frac{x}{2} = \frac{1-\cos x}{\sin x} = \frac{1-\left(\frac{-\sqrt{21}}{5}\right)}{2/5} = \boxed{\frac{5+\sqrt{21}}{2}}$$

LCD = 5

$\cos y = \frac{-2}{5}$, y is in QIII, Find $\sin 2y$.

$180^\circ < y < 270^\circ$



$$\sin 2y = 2 \sin y \cos y$$

$$= 2 \cdot \frac{-\sqrt{21}}{5} \cdot \frac{-2}{5}$$

$$= \boxed{\frac{4\sqrt{21}}{25}}$$

$\cos 2x = 2\cos^2 x - 1 \Rightarrow \boxed{\cos^2 x = \frac{1 + \cos 2x}{2}}$

$\cos 2x = 1 - 2\sin^2 x \Rightarrow \boxed{\sin^2 x = \frac{1 - \cos 2x}{2}}$

Prove $\boxed{\tan^2 x = \frac{1 - \cos 2x}{1 + \cos 2x}}$

Lowering Powers

$$\tan^2 x = \frac{\sin^2 x}{\cos^2 x} = \frac{\frac{1 - \cos 2x}{2}}{\frac{1 + \cos 2x}{2}} = \frac{1 - \cos 2x}{1 + \cos 2x}$$

LCD = 2

$\sin(x+y) = \sin x \cos y + \cos x \sin y$

$\sin(x-y) = \sin x \cos y - \cos x \sin y$

Add them

$\sin(x+y) + \sin(x-y) = 2\sin x \cos y$

Divide by 2

$\sin x \cos y = \frac{1}{2}\left[\sin(x+y) + \sin(x-y)\right]$

Product - to - Sum Formulas

$\sin u \cos v = \frac{1}{2}\left[\sin(u+v) + \sin(u-v)\right]$

$\cos u \sin v = \frac{1}{2}\left[\sin(u+v) - \sin(u-v)\right]$

$\cos u \cos v = \frac{1}{2}\left[\cos(u+v) + \cos(u-v)\right]$

$\sin u \sin v = \frac{1}{2}\left[\cos(u-v) - \cos(u+v)\right]$

$\sin 3x \sin 5x = \frac{1}{2}\left[\cos(3x-5x) - \cos(3x+5x)\right]$

$= \frac{1}{2}\left[\cos(-2x) - \cos 8x\right]$

$= \frac{1}{2}\left[\cos 2x - \cos 8x\right]$

Recall

$\cos(-\alpha) = \cos\alpha$

write $4\cos 4x \cos 8x$ as product-to-Sum

$$4\cos 4x \cos 8x = 4 \cdot \frac{1}{2}\left[\cos(4x+8x) + \cos(4x-8x)\right]$$

$$= 2\left[\cos 12x + \cos(-4x)\right]$$

$\cos(-\alpha) = \cos\alpha$

$$= \boxed{2\left[\cos 12x + \cos 4x\right]}$$

Find the exact Value of

$$\cos 37.5^\circ \sin 7.5^\circ = \frac{1}{2}\left[\sin(37.5^\circ + 7.5^\circ) - \sin(37.5^\circ - 7.5^\circ)\right]$$

$$= \frac{1}{2}\left[\sin 45^\circ - \sin 30^\circ\right]$$

$$= \frac{1}{2}\left[\frac{\sqrt{2}}{2} - \frac{1}{2}\right] = \frac{1}{2} \cdot \frac{\sqrt{2}-1}{2}$$

$$= \boxed{\frac{\sqrt{2}-1}{4}}$$

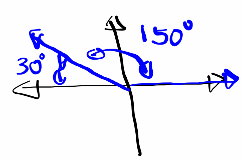
Find the exact Value of

$2\sin 52.5^\circ \sin 97.5^\circ$

$$\sin u \sin v = \frac{1}{2}\left[\cos(u-v) - \cos(u+v)\right]$$

$$2\sin 52.5^\circ \sin 97.5^\circ = 2 \cdot \frac{1}{2}\left[\cos(52.5^\circ - 97.5^\circ) - \cos(52.5^\circ + 97.5^\circ)\right]$$

$$= \cos(-45^\circ) - \cos 150^\circ$$



$$= \cos 45^\circ - (-\cos 30^\circ)$$

$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} = \boxed{\frac{\sqrt{2}+\sqrt{3}}{2}}$$

Product-to-Sum Formulas

$\sin u \cos v = \frac{1}{2}[\sin(u+v) + \sin(u-v)]$

$\cos u \sin v = \frac{1}{2}[\sin(u+v) - \sin(u-v)]$

$\cos u \cos v = \frac{1}{2}[\cos(u+v) + \cos(u-v)]$

$\sin u \sin v = \frac{1}{2}[\cos(u-v) - \cos(u+v)]$

What about Sum-to-Product?

✓ $\sin x + \sin y = 2 \sin\frac{x+y}{2} \cos\frac{x-y}{2}$

$\sin x - \sin y = 2 \cos\frac{x+y}{2} \sin\frac{x-y}{2}$

$\cos x + \cos y = 2 \cos\frac{x+y}{2} \cos\frac{x-y}{2}$

$\cos x - \cos y = -2 \sin\frac{x+y}{2} \sin\frac{x-y}{2}$

$\sin u \cos v = \frac{1}{2}[\sin(u+v) + \sin(u-v)]$

Let $u = \frac{x+y}{2}$ and $v = \frac{x-y}{2}$

$u+v = \frac{x+y}{2} + \frac{x-y}{2} = \frac{x+y+x-y}{2} = \frac{2x}{2} = x$

$u-v = \frac{x+y}{2} - \frac{x-y}{2} = \frac{x+y-(x-y)}{2} = \frac{\overbrace{x+y-x+y}^{2y}}{2} = y$

$\sin u \cos v = \frac{1}{2}[\sin(u+v) + \sin(u-v)]$

$\sin\frac{x+y}{2} \cos\frac{x-y}{2} = \frac{1}{2}[\sin x + \sin y]$

Multiply by 2

$$\boxed{2 \sin\frac{x+y}{2} \cos\frac{x-y}{2} = \sin x + \sin y}$$

Write $\sin 3x + \sin 7x$ as product.

$\sin 3x + \sin 7x = 2 \sin\frac{3x+7x}{2} \cos\frac{3x-7x}{2}$

$= 2 \sin 5x \cos(-2x)$

Recall

$\cos(-\alpha) = \cos\alpha$

$= \boxed{2 \sin 5x \cos 2x}$

Write $\cos 10x - \cos 2x$ as product.

$$\cos x - \cos y = -2 \sin \frac{x+y}{2} \sin \frac{x-y}{2}$$

$$\cos 10x - \cos 2x = -2 \sin \frac{10x+2x}{2} \sin \frac{10x-2x}{2}$$

$$= \boxed{-2 \sin 6x \sin 4x}$$

Find the exact value of

$$\sin 75^\circ + \sin 15^\circ = 2 \sin \frac{75^\circ+15^\circ}{2} \cos \frac{75^\circ-15^\circ}{2}$$

$$\sin x + \sin y = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}$$

$$= 2 \sin 45^\circ \cdot \cos 30^\circ$$

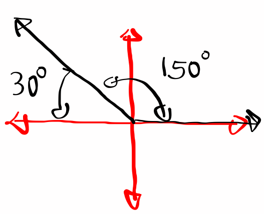
$$= \cancel{2} \cdot \frac{\sqrt{2}}{\cancel{2}} \cdot \frac{\sqrt{3}}{2} = \boxed{\frac{\sqrt{6}}{2}}$$

Show $\cos 100^\circ - \cos 200^\circ = \sin 50^\circ$ ✓

$$\cos x - \cos y = -2 \sin \frac{x+y}{2} \sin \frac{x-y}{2}$$

$$\cos 100^\circ - \cos 200^\circ = -2 \sin \frac{100^\circ+200^\circ}{2} \sin \frac{100^\circ-200^\circ}{2}$$

$$= -2 \sin 150^\circ \underbrace{\sin(-50^\circ)}$$



$$= \cancel{-2} \cdot \frac{1}{\cancel{2}} \cdot (-\sin 50^\circ)$$

$$= \sin 50^\circ$$

$\sin 150^\circ = \sin 30^\circ = \frac{1}{2}$

Recall

$\sin(-\alpha) = -\sin \alpha$

Verify

$$\frac{\sin x + \sin y}{\cos x + \cos y} = \tan\left(\frac{x+y}{2}\right)$$

$$\frac{\sin x + \sin y}{\cos x + \cos y} = \frac{\cancel{2} \cdot \sin\frac{x+y}{2} \cdot \cancel{\cos\frac{x-y}{2}}}{\cancel{2} \cdot \cos\frac{x+y}{2} \cdot \cancel{\cos\frac{x-y}{2}}}$$

$$= \frac{\sin\frac{x+y}{2}}{\cos\frac{x+y}{2}} = \tan\left(\frac{x+y}{2}\right)$$

Verify

$$\frac{\sin x + \sin 2x + \sin 3x + \sin 4x + \sin 5x}{\boxed{\cos x} + \boxed{\cos 2x} + \cos 3x + \boxed{\cos 4x} + \boxed{\cos 5x}} = \tan 3x$$

$$\sin x + \sin 5x = 2\sin\frac{x+5x}{2}\cos\frac{x-5x}{2} = 2\sin 3x\cos 2x$$

$$\sin 2x + \sin 4x = 2\sin\frac{2x+4x}{2}\cos\frac{2x-4x}{2} = 2\sin 3x\cos x$$

$$\frac{2\sin 3x\cos 2x + 2\sin 3x\cos x + \sin 3x}{\boxed{2\cos 3x\cos 2x} + \boxed{2\cos 3x\cos x} + \cos 3x} = \frac{\sin 3x\,\cancel{[2\cos 2x + 2\cos x + 1]}}{\cos 3x\,\cancel{[2\cos 2x + 2\cos x + 1]}}$$

$$= \boxed{\tan 3x}$$

$$\cos x + \cos 5x = 2\cos\frac{x+5x}{2}\cos\frac{x-5x}{2} = 2\cos 3x\cos 2x$$

$$\cos 2x + \cos 4x = 2\cos\frac{2x+4x}{2}\cos\frac{2x-4x}{2} = 2\cos 3x\cos x$$

Verify $\dfrac{\sin 10x}{\sin 9x + \sin x} = \dfrac{\cos 5x}{\cos 4x}$

Hint: $10x = 2 \cdot 5x$

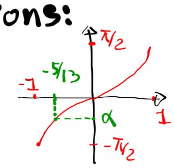
$$\frac{\sin 10x = \sin 2(5x) = 2 \sin 5x \cos 5x}{\sin 9x + \sin x = 2 \sin \frac{9x+x}{2} \cos \frac{9x-x}{2} = 2 \sin 5x \cos 4x}$$

$$\frac{\sin 10x}{\sin 9x + \sin x} = \frac{\cancel{2}\, \cancel{\sin 5x}\, \cos 5x}{\cancel{2}\, \cancel{\sin 5x}\, \cos 4x} = \frac{\cos 5x}{\cos 4x}$$

More on inverse functions:

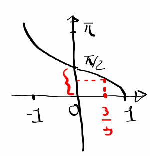
$y = \sin^{-1} x \iff x = \sin y$

$-1 \le x \le 1 \quad , \quad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$



$y = \cos^{-1} x \iff x = \cos y$

$-1 \le x \le 1 \quad , \quad 0 \le y \le \pi$



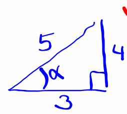
$y = \tan^{-1} x \iff x = \tan y$

$-\infty < x < \infty \quad , \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$

Find $\sin\left(2 \cos^{-1} \frac{3}{5}\right) = \sin 2\alpha$

Let $\alpha = \cos^{-1} \frac{3}{5}$ $\qquad = 2 \sin\alpha \cos\alpha$

$\cos \alpha = \frac{3}{5}$ → QI $\qquad = 2 \cdot \frac{4}{5} \cdot \frac{3}{5}$

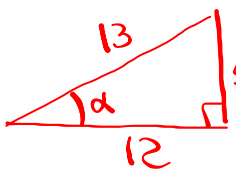


$= \boxed{\frac{24}{25}}$

Find $\cos\left(\frac{1}{2}\sin^{-1}\frac{-5}{13}\right) = \cos\frac{1}{2}\alpha$

Let $\alpha = \sin^{-1}\frac{-5}{13}$

$\sin\alpha = \frac{-5}{13}$ → QIV



$-90° < \alpha < 0°$

$-45° < \frac{\alpha}{2} < 0°$

$\alpha/2$ is in QIV

$= \cos\frac{\alpha}{2}$

$= \pm\sqrt{\frac{1+\cos\alpha}{2}}$

$= +\sqrt{\frac{1+12/13}{2}}$ LCD=13

$= \sqrt{\frac{13+12}{26}}$

$= \sqrt{\frac{25}{26}} = \frac{5}{\sqrt{26}}\cdot\frac{\sqrt{26}}{\sqrt{26}} = \boxed{\frac{5\sqrt{26}}{26}}$

Find exact Value of

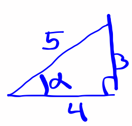
$\tan\left(\sin^{-1}\frac{3}{5} - \sin^{-1}\frac{5}{13}\right) = \tan(\alpha-\beta)$

Let $\alpha = \sin^{-1}\frac{3}{5}$

$\beta = \sin^{-1}\frac{5}{13}$

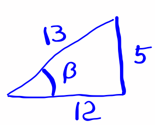
$\sin\alpha = \frac{3}{5}$

QI



$\sin\beta = \frac{5}{13}$

QI

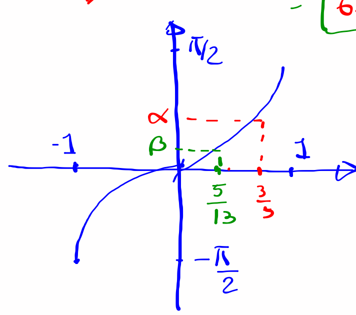


$= \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

$= \frac{\frac{3}{4} - \frac{5}{12}}{1 + \frac{3}{4}\cdot\frac{5}{12}}$

LCD=48

$= \boxed{\frac{16}{63}}$



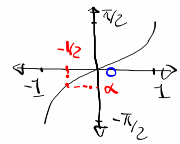
Find exact value of

$\cos\left(\underbrace{\sin^{-1}\tfrac{-1}{2}}_{\alpha} - \underbrace{\tan^{-1}(-1)}_{\beta}\right) = \cos(\alpha - \beta)$

$\alpha = \sin^{-1}\frac{-1}{2}$

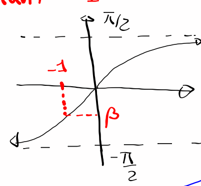
QIV

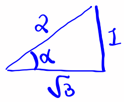
$\sin\alpha = \frac{-1}{2}$



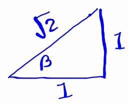
$\beta = \tan^{-1}(-1)$ QIV

$\tan\beta = -1$





$\alpha = -30°$



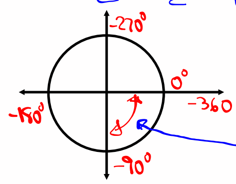
$\beta = -45°$

$\cos(-30° - -45°)$

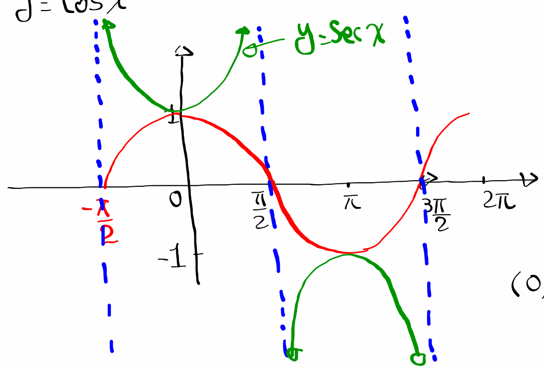
$= \cos 15°$

$\cos\alpha\cos\beta + \sin\alpha\sin\beta$

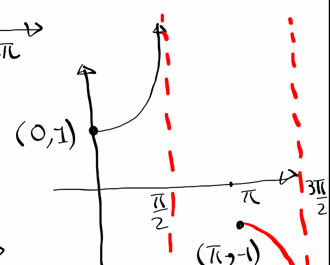
$= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} + \frac{-1}{2}\cdot\frac{-\sqrt{2}}{2} = \boxed{\frac{\sqrt{6}+\sqrt{2}}{4}} = \boxed{?}$



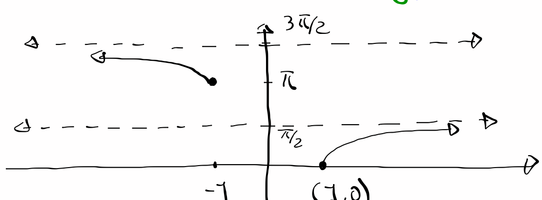
$y = \cos x$



$0 \le x < \frac{\pi}{2}$



$\pi \le x < 3\pi/2$



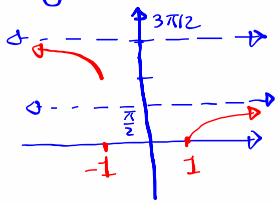
$y = \sec^{-1} x$

$x \le -1$ OR $x \ge 1$

$\pi \le y < \frac{3\pi}{2}$ OR $0 \le y < \pi$

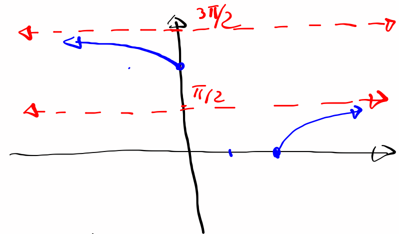
Graph

$y = \sec^{-1}(x-1)$

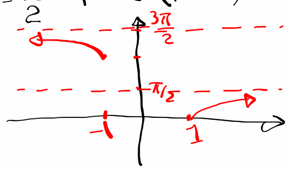


$x-1 \le -1$ $\quad$ $x-1 \ge 1$

$x \le 0$ $\quad$ $x \ge 2$

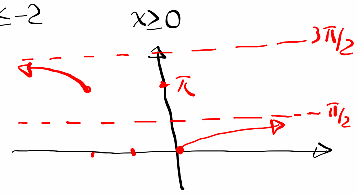


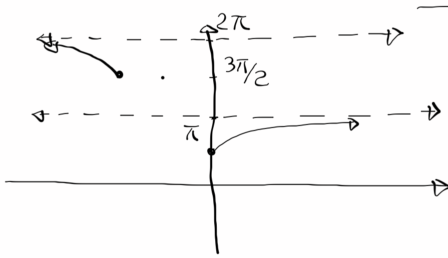
$y = \frac{\pi}{2} + \sec^{-1}(x+1)$



$x+1 \le -1$ $\quad$ $x+1 \ge 1$

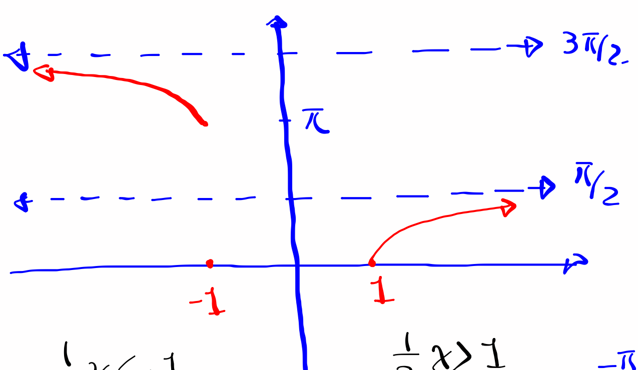
$x \le -2$ $\quad$ $x \ge 0$





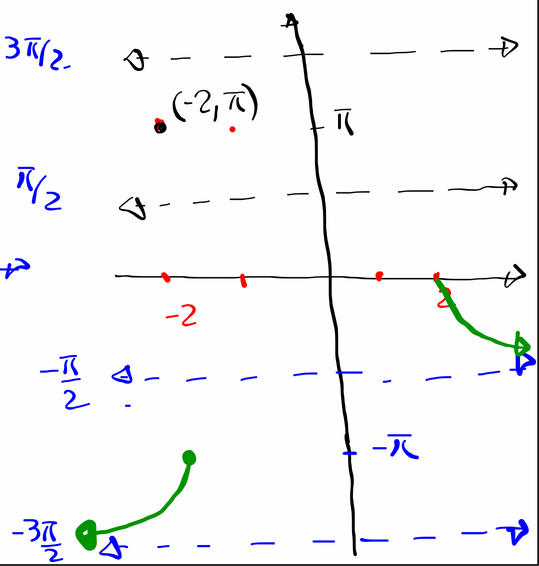
Graph $y = -\sec^{-1}\left(\frac{1}{2}x\right)$

Reflection about x-axis



$\frac{1}{2}x \le -1$ $\quad$ $\frac{1}{2}x \ge 1$

$x \le -2$ $\quad$ $x \ge 2$



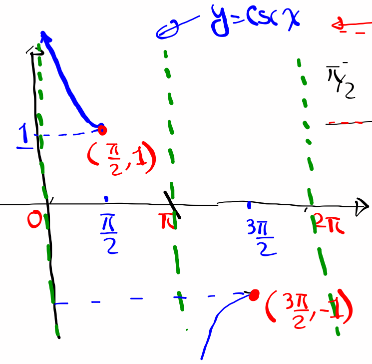
$y = \sin x$

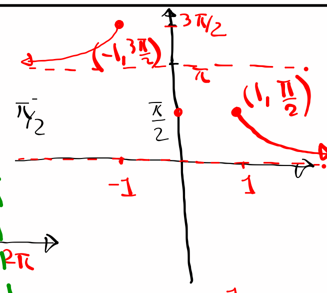
$0 < x \le \frac{\pi}{2}$

$y \ge 1$

$\pi < x \le \frac{3\pi}{2}$

$y \le -1$



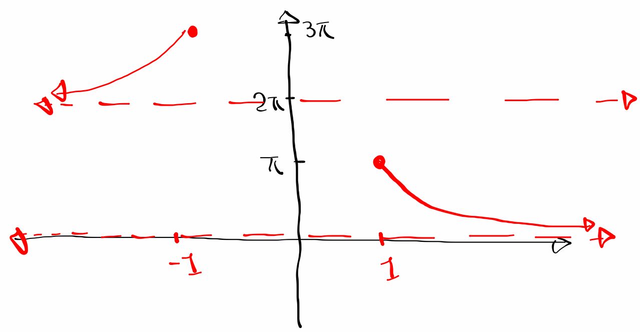


$y = \csc^{-1} x$

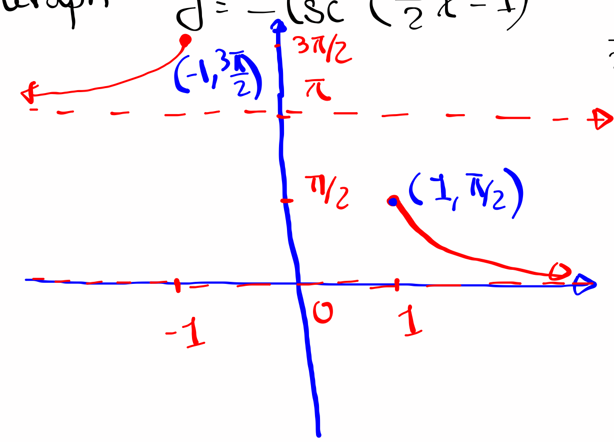
$x \ge 1$ or $x \le -1$

$0 < y \le \frac{\pi}{2}$, $\pi < y \le \frac{3\pi}{2}$

Graph $y = 2\csc^{-1} x$

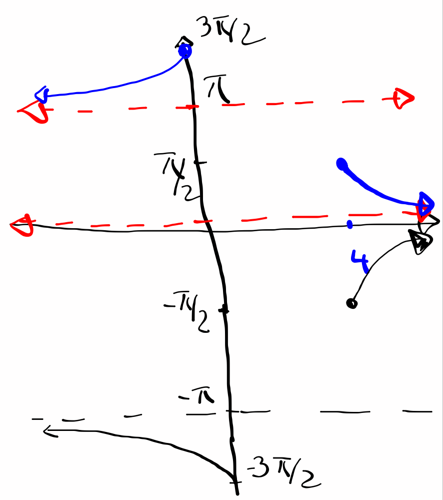


Graph $y = -\csc^{-1}(\frac{1}{2}x - 1)$



$\frac{1}{2}x - 1 \le -1$ $\qquad$ $\frac{1}{2}x - 1 \ge 1$

$x \le 0$ $\qquad$ $x \ge 4$

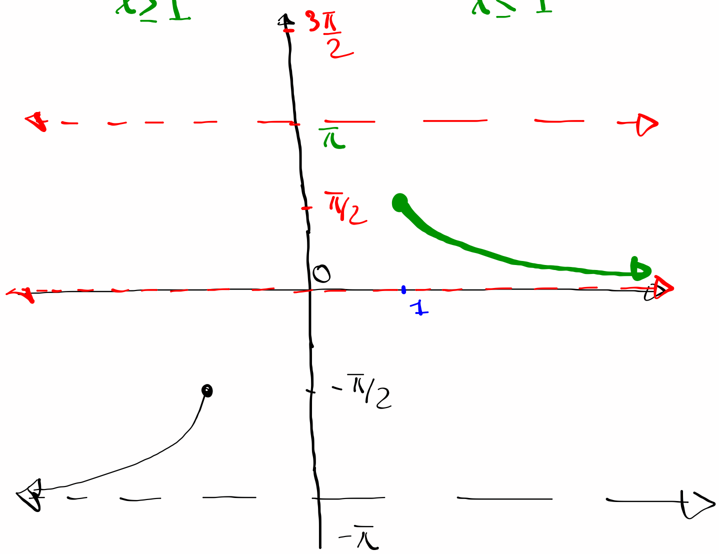


Graph $y = -\pi + \csc^{-1}(-x)$

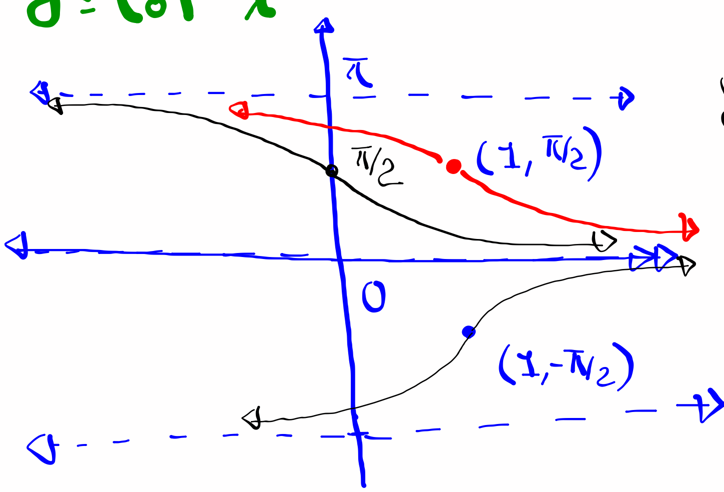
$-x \le -1$ OR $-x \ge 1$

x	y
1	
-1	

$x \ge 1$ $x \le -1$



$y = \cot^{-1} x$



Graph

$y = -\cot^{-1}(x-1)$

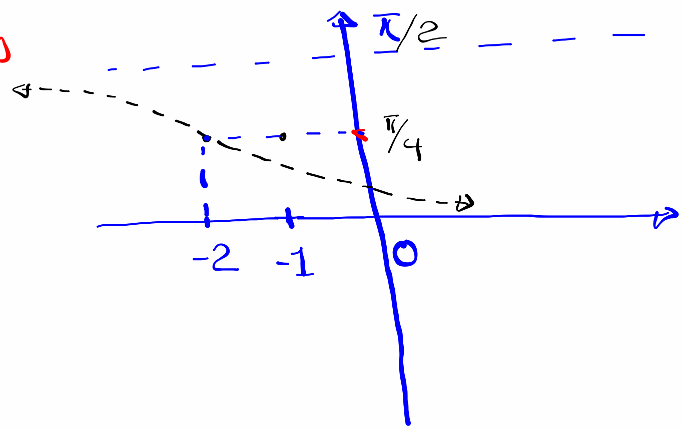
Shift Right 1

Reflection about x-axis

Graph $y = \frac{1}{2}\cot^{-1}(x+2)$

Shift left 2 units

Compress vertically by $\frac{1}{2}$ factor



$\cot ? = 0$

$\frac{\cos ?}{\sin ?} = 0 \qquad \cos ? = 0 \qquad \frac{\pi}{2}$